

ON SEMILOCALLY SIMPLY CONNECTED SPACES

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ABSTRACT. The purpose of this paper is: (i) to construct a space which is semilocally simply connected in the sense of Spanier even though its Spanier group is non-trivial; (ii) to propose a modification of the notion of a Spanier group so that via the modified Spanier group semilocal simple connectivity can be characterized; and (iii) to point out that with just a slightly modified definition of semilocal simple connectivity which is sometimes also used in literature, the classical Spanier group gives the correct characterization within the general class of path-connected topological spaces.

While the condition “semilocally simply connected” plays a crucial role in classical covering theory, in generalized covering theory one needs to consider the condition “homotopically Hausdorff” instead. The paper also discusses which implications hold between all of the abovementioned conditions and, via the modified Spanier groups, it also unveils the weakest so far known algebraic characterization for the existence of generalized covering spaces as introduced by Fischer-Zastrow. For most of the implications, the paper also proves the non-reversibility by providing the corresponding examples. Some of them rely on spaces that are newly constructed in this paper.

1. INTRODUCTION

This paper was motivated by an observation during the research of [6], namely that E.H. Spanier, when writing his celebrated book on algebraic topology [12], apparently made an oversight in the statement which immediately precedes Corollary 2.5.14. That statement, in which he characterizes semilocal simple connectivity in terms of vanishing of a certain group $\pi(\mathcal{U}, x_0)$ for at least one open covering \mathcal{U} of the space, turns out to be correct only if one additionally assumes local path-connectedness. Of course, one may wonder if this assumption was perhaps not implicitly made. However, in view of the author’s great attention to details in this book in general, our ultimate conclusion was that Spanier would have mentioned this additional assumption had he been aware of the phenomena and examples that we shall expose below.

Roughly speaking, a problem occurs for spaces which are not locally path-connected. This is because the fundamental group uses base points, whereas the subgroup of the fundamental group which Spanier associated with a covering (and which we henceforth call the *Spanier group*) does not use base points in a similar way. Of course, this group is defined as a subgroup of the fundamental group and therefore it depends on the same base point as the fundamental group, but the concept of Spanier groups refrains from using base points for each set of the covering separately. Therefore Spanier’s characterization of semilocal simple connectivity (cf. immediately before Corollary 2.5.14) matches his definition (cf. immediately before Theorem 2.4.10) only in the locally path-connected case.

The main purpose of our paper is to:

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- confirm this assertion by constructing a space Y which is semilocally simply connected in the sense of Spanier, but its Spanier group is non-trivial (cf. Proposition 3.1);
- confirm the “if”-part of the claim preceding Corollary 2.5.14, for all spaces and the “only-if” part for locally path-connected spaces (Theorem 2.8(3)–(4));
- propose a modification of Spanier groups so that the corresponding claim will be correct for all spaces (Theorem 2.8(2)) and
- propose a modification of the definition of semilocal simple connectivity and prove that with this modified definition and with the original definition of Spanier groups the claim preceding Corollary 2.5.14 is also correct (Theorem 2.8(1)).

Accordingly, there will be two concepts of semilocal simple connectivity and two versions of Spanier groups — one which depends on base points and one which does not. In order to avoid ambiguity in the terminology of the present paper, we will from now on speak of these concepts using the attributes “based” and “unbased” (cf. Definitions 2.1–2.3 and 2.5).

While semilocal simple connectivity is a crucial condition in classical covering space theory, the generalized covering space theory treated in [6] mainly considered the weaker condition called “homotopically Hausdorff”. The paper [3] also studied two versions of this condition, one which depends on base points and one which does not. However, in our case we will adopt the notation from [3] and use the attributes “weak” and “strong” (cf. Definition 2.10), respectively. When base points are treated correctly, the conditions of semilocal simple connectivity can be equivalently described by properties of Spanier groups (Theorem 2.8(1)–(2)). However, we only know sufficient conditions on Spanier groups which imply homotopic Hausdorffness (Section 6, (11)–(12)). In this context it should also be pointed out that even the weakest of these conditions for Spanier groups implies a condition that we will call “homotopically path-Hausdorff”. This condition was not mentioned in [3] or [6], but is similar to a condition that appeared under a different name in [14], a preprint preceding [6]. Therefore the concept of based Spanier groups, introduced in this paper, also apparently yields the weakest currently known algebraic sufficient condition for the existence of generalized universal covering spaces (Theorem 2.9).

We also wish to point out that we are not aware of the above described incorrectness in Spanier’s book leading to a false theorem therein. All places that we found, where the crucial remark preceding Corollary 2.5.14 has been applied, were statements where local path-connectedness of the underlying topological space has been an assumption, and in this framework the crucial statement is correct.

Our paper will in Section 6 also briefly discuss known implications among all the mathematical concepts mentioned so far. All topological spaces in this article are assumed to be path-connected. Since we need to consider the fundamental group, it does not seem necessary to consider more general spaces.

2. DEFINITIONS AND TERMINOLOGY

Definition 2.1. *We call a topological space X (based) semilocally simply connected if for every point $x \in X$ there exists a neighbourhood U of x such that the inclusion-induced homomorphism $\pi_1(U, x) \rightarrow \pi_1(X, x)$ is trivial.*

The majority of topology books discussing covering spaces seems to prefer this definition (see e.g. p. 63 of [8], p. 393 of [11], p. 174 of [10] and p. 78 of [12]) over the

following; we will only use the attribute “based” in connection with semilocal simple connectivity where it is needed to distinguish this definition from the following.

Definition 2.2. *We call a topological space X unbased semilocally simply connected if for every point $x \in X$ there exists a neighbourhood U of x such that every loop in U is null-homotopic in X .*

The latter definition is used, for example, in [7, p. 187], and, named slightly differently, in [9, Definition 6.6.8, p. 255].

Definition 2.3. *Let X be a space, $x_0 \in X$ a base point, and $\mathcal{U} = \{U_i \mid i \in I\}$ an arbitrary open covering of X . Then we define $\pi(\mathcal{U}, x_0)$ to be the subgroup of $\pi_1(X, x_0)$ which contains all homotopy classes having representatives of the following type:*

$$(2.3) \quad \prod_{j=1}^n u_j v_j u_j^{-1},$$

where u_j are arbitrary paths (starting at the base point x_0) and each v_j is a loop inside one of the neighbourhoods $U_i \in \mathcal{U}$. We call this group the (unbased) Spanier group with respect to \mathcal{U} .

This definition matches the definition from [12, Chapter 2, Section 5 between items 7 and 8]. In [12] this notation was already used, but names have not yet been given to these groups.

We choose the name *Spanier groups*, since all traces in the literature that we are aware of seem to go back to this appearance in Spanier’s book.

In the introduction we announced a concept that introduces base points to the sets of open coverings. Consequently, we will instead of open sets U also consider “pointed open sets”, i.e. pairs (U, x) , where $x \in U$ and U is open.

Definition 2.4. *Let X be a space.*

(1) *An open covering of X by pointed sets is a family of pointed open sets $\mathcal{V} = \{(U_i, x_i) \mid i \in I\}$, where*

$$(2.4) \quad \{x_i \mid i \in I\} = X.$$

(2) *Refinements between coverings by pointed sets are defined as follows: $\mathcal{U}' = \{(U'_i, x'_i) \mid i \in I\}$ refines $\mathcal{U} = \{(U_j, x_j) \mid j \in J\}$, if $\forall i \exists j$ such that $U'_i \subset U_j$ and $x_i = x_j$.*

Let $\mathcal{U} = \{U_i \mid i \in I\}$ be a covering of X by open sets. Observe that due to expression (2.4) demanding that each point of X occurs at least once as base point of one of the covering sets, it will in general not suffice to choose a base point for each of the U_i in order to turn it into an open covering \mathcal{V} of X by pointed sets. Instead, the following procedure is apparently in general necessary:

- for each $U_i \in \mathcal{U}$ take $|U_i|$ copies into \mathcal{V} ; and
- define each of those copies as (U_i, P) , i.e. use the same set U_i as first entry, and let the second entry run over all points $P \in U_i$.

When constructed with this procedure, coverings by neighbourhood pairs offer in principle the same options for refinements as coverings by open sets. Vice versa note, that this procedure will usually generate such coverings by pointed sets, where a lot of $x \in X$ occur as base points for different sets U_i .

Definition 2.5. *Let X be a space, $x_0 \in X$, and $\mathcal{V} = \{(U_i, x_i) \mid i \in I\}$ be a covering of X by open neighbourhood pairs. Then we let $\pi^*(\mathcal{V}, x_0)$ be the subgroup of*

$\pi_1(X, x_0)$ which contains all homotopy classes having representatives of the following type:

$$(2.5) \quad \prod_{j=1}^n u_j \ v_j \ u_j^{-1},$$

where the u_j are arbitrary paths that run from x_0 to some point x_i and each v_j then must be a closed path inside the corresponding U_i . We will call this group the based Spanier group with respect to \mathcal{V} .

Remark 2.6. Note that for based and unbased Spanier groups the following holds: Let \mathcal{U}, \mathcal{V} be open coverings, and let \mathcal{U} be a refinement of \mathcal{V} . Then $\pi(\mathcal{U}) \subset \pi(\mathcal{V})$. Analogously $\pi^*(\mathcal{U}) \subset \pi^*(\mathcal{V})$ holds, assuming that \mathcal{U} and \mathcal{V} are now open coverings by pointed sets. Due to these inclusion relations, there exist inverse limits of these Spanier groups, defined via the directed system of all coverings with respect to refinement. We will call them the (unbased) Spanier group and the based Spanier group of the space X and denote them by

$$\varprojlim(\pi(\mathcal{U})) \text{ and } \varprojlim(\pi^*(\mathcal{V})),$$

respectively, observing that these inverse limits are realized by intersections:

$$\bigcap_{\text{coverings } \mathcal{U}} \pi(\mathcal{U}) = \varprojlim(\pi(\mathcal{U})) \text{ and } \bigcap_{\substack{\text{coverings } \mathcal{V} \\ \text{by pointed sets}}} \pi^*(\mathcal{V}) = \varprojlim(\pi^*(\mathcal{V})).$$

Since for locally path-connected spaces any covering refines to a covering by path-connected sets, the based and unbased Spanier groups coincide in that case.

Remark 2.7. Recall that all topological spaces in this paper are assumed to be path-connected. Despite discussing the properties “based” and “unbased”, by definition the fundamental group and (since they are defined as subgroups) also all Spanier groups formally depend on a base point. However, the standard argument that the isomorphism type and many other essential properties of the fundamental group of a path-connected space do indeed not depend on the base point, naturally extends to Spanier groups. We will therefore omit the base point in our notation, whenever appropriate.

The following two theorems are our main results, and Sections 4 and 5, respectively, are devoted to prove them.

Theorem 2.8.

- (1) Let X be an arbitrary space, $x_0 \in X$. Then X is unbased semilocally simply connected, if and only if it has an open covering \mathcal{U} such that $\pi(\mathcal{U}, x_0)$ is trivial.
- (2) Let X be an arbitrary space, $x_0 \in X$. Then X is semilocally simply connected, if and only if it has an open covering \mathcal{V} by pointed sets such that $\pi^*(\mathcal{V}, x_0)$ is trivial.
- (3) Let X be an arbitrary space. Then the two equivalent properties from (1) imply those from (2).
- (4) Let X be a locally path-connected space. Then the two equivalent properties from (2) imply those from (1).
- (5) For topological spaces that are not locally path-connected, the properties from (2) need not imply those from (1). The space Y that will be constructed in Section 3, satisfies (2) but not (1).

Theorem 2.9. Let X be a topological space whose based Spanier group $\varprojlim \pi^*(\mathcal{U})$ is trivial. Then X admits a generalized universal covering space in the sense of [6].

With items (1) and (2) of the next definition we follow [3, p. 1091]:

Definition 2.10.

- (1) *A space X is called (weakly) homotopically Hausdorff if for every $x_0 \in X$ and for every non-trivial $\alpha \in \pi_1(X, x_0)$ there exists a neighbourhood U of x_0 such that no loop in U is homotopic (in X) to α rel. x_0 . An equivalent condition, using the terminology of [13, Definition 1], would be the absence of non-trivial small loops.*
- (2) *A space X is called strongly homotopically Hausdorff, if for every $x_0 \in X$ and for every essential closed curve γ in X there is a neighbourhood of x_0 that contains no closed curve freely homotopic (in X) to γ .*
- (3) *A space X is called homotopically path-Hausdorff, provided that it satisfies the following property with respect to any two paths $w, v : [0, 1] \rightarrow X$ with $w(0) = v(0)$ and $w(1) = v(1)$: If w and v are not homotopic relative to the endpoints, then there exist $0 = t_0 < t_1 < \dots < t_k = 1$ and open sets U_1, U_2, \dots, U_k with $w([t_{i-1}, t_i]) \subseteq U_i$ for $1 \leq i \leq k$ such that for any $w_i : [t_{i-1}, t_i] \rightarrow U_i$ with $w_i(t_{i-1}) = w(t_{i-1})$ and $w_i(t_i) = w(t_i)$, the concatenation $w_1 * w_2 * \dots * w_k$ is not homotopic to v relative to the endpoints.*

The terms weakly and strongly homotopically Hausdorff already appeared, but under different names, in [14, 1.1], while the third property that was considered there, although similar to our Definition 2.10(3), is not equivalent to it. While ‘strongly homotopically Hausdorff’ appeared for the first time in peer-reviewed literature in [3], ‘homotopically Hausdorff’, had been used before in [2, Definition 5.2].

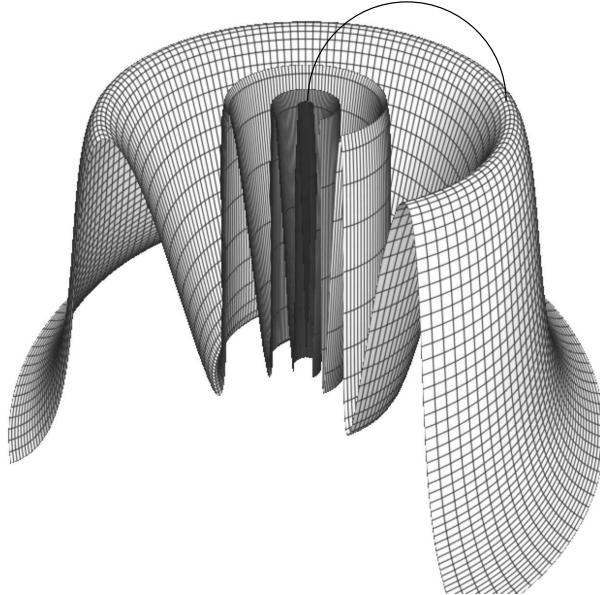
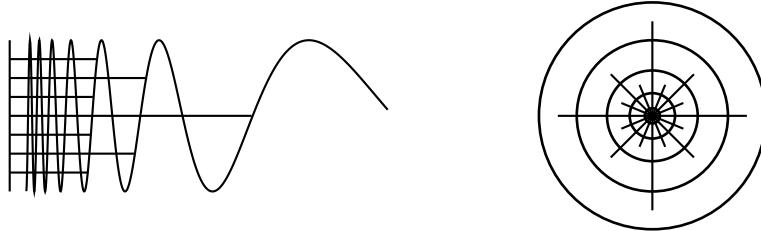
Remark 2.11. Note that all strongly homotopically Hausdorff spaces and all homotopically path-Hausdorff spaces are homotopically Hausdorff. *If we apply the condition of homotopical path-Hausdorffness to a constant path at x we obtain the condition for weak homotopical Hausdorffness at x ; thus the second statement. The first statement directly follows from the definitions.*

3. EXAMPLES

The space Y' : One of our examples $Y' \subset \mathbb{R}^3$ will be precisely the space called A in [3], and was defined at the beginning of Section 3 therein. It consists of a rotated topologists’ sine curve (as suggested by Figure 1), the “central axis”, where this surface tends to, and a system of horizontal arcs is attached to them so that they become dense (only) near the central axis. Figure 2 shows schematically the top-view and the side-view of defining such arcs, which we shall call “tunnels”. Their presence is indicated by the prime of Y' .

The space Y : Another important example will be our space $Y \subset \mathbb{R}^3$ (see Figure 1). It consists of the same surface portion as Y' , the central axis to which this surface portion tends, but instead of defining a system of arcs for connecting central axis and surface portion, we just connect them by a single arc C . This arc C can be easily embedded into \mathbb{R}^3 , so as not to intersect the surface portion or the central axis at any other points than its endpoints.

The space Z' : Our third example will be called $Z' \subset \mathbb{R}^3$. It is precisely the space called B in [3] and defined immediately before Theorem 3.4 therein. It consists of a rotated topologists’ sine curve (as shown in Figure 3), the “outer cylinder” at radius 1, where this surface tends to, and a system of horizontal arcs attached to them so that they become dense (only) near the outer cylinder. Figure 4 shows schematically the top-view and the side-view of how to define such arcs. We will call these arcs “tunnels” as well.

FIGURE 1. The space Y .FIGURE 2. The radial projections and the top view of the tunnels of Y' .

The space Z : Analogously, we will also need a space which has the same outer cylinder and surface portion as Z' , but where outer cylinder and surface portion are just connected by a single arc C (similar as for Y), but which has no tunnel-system. We will call this space Z .

Apart from these spaces, we will also need the *Hawaiian Earring*, that we will denote by HE . This is a more well-known space, it is a countable union of circles in the plane, as pictured in Figure 5.

Proposition 3.1. *The space Y has the following properties:*

- (1) *Its Spanier group is non-trivial;*
- (2) *its based Spanier group is trivial;*
- (3) *it is semilocally simply connected;*
- (4) *it is not unbased semilocally simply connected;*
- (5) *it is homotopically Hausdorff;*
- (6) *it is not strongly homotopically Hausdorff;*
- (7) *it is homotopically path-Hausdorff.*

Proposition 3.2. *The space Y' has the following properties:*

- (1) *Its Spanier group is non-trivial;* 1

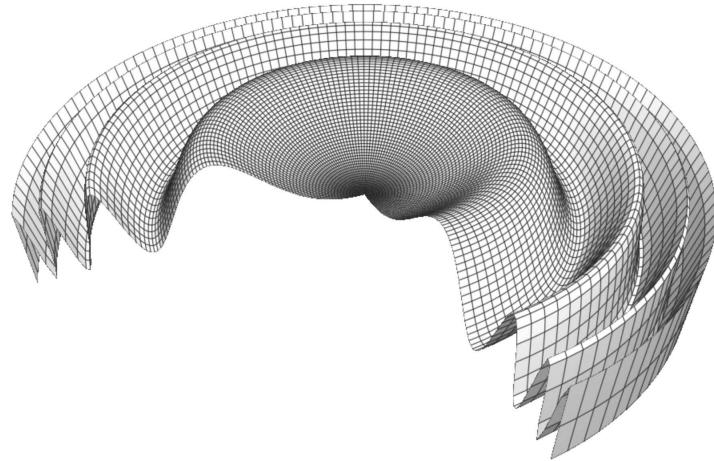
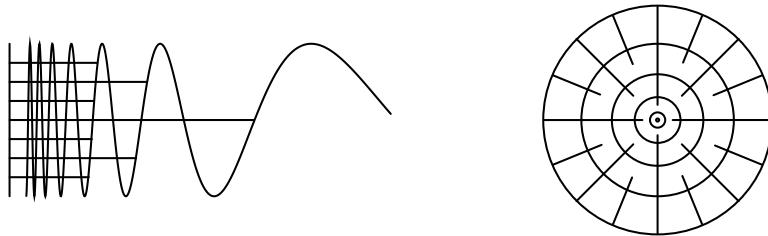
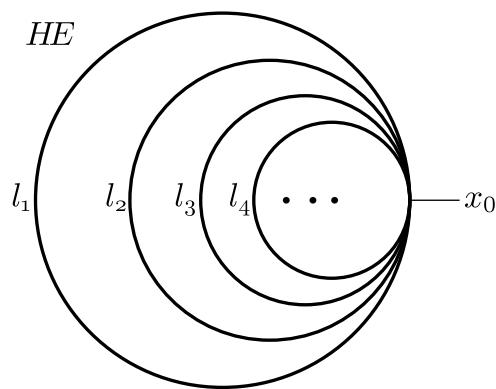
FIGURE 3. The “surface” portion of the spaces Z and Z' .FIGURE 4. The radial projections and the top view of the tunnels of Z' .

FIGURE 5. The Hawaiian Earring

- (2) its based Spanier group is non-trivial;
- (3) it is not semilocally simply connected;
- (4) it is not unbased semilocally simply connected;
- (5) it is homotopically Hausdorff;
- (6) it is not strongly homotopically Hausdorff;
- (7) it is homotopically path-Hausdorff.

Proposition 3.3. *The space Z has the following properties:*

- (1) Its Spanier group is trivial;
- (2) its based Spanier group is trivial;
- (3) it is semilocally simply connected;
- (4) it is unbased semilocally simply connected;
- (5) it is homotopically Hausdorff;
- (6) it is strongly homotopically Hausdorff;
- (7) it is homotopically path-Hausdorff.

Proposition 3.4. *The space Z' has the following properties:*

- (1) Its Spanier group is non-trivial;
- (2) its based Spanier group is non-trivial;
- (3) it is not semilocally simply connected;
- (4) it is not unbased semilocally simply connected;
- (5) it is homotopically Hausdorff;
- (6) it is strongly homotopically Hausdorff;
- (7) it is not homotopically path-Hausdorff.

Convention: Given any path α defined on $[0, 1]$, we define a path α^{-1} by $\alpha^{-1}(t) := \alpha(1 - t)$.

Proof of Proposition 3.1. (1) Fix a point x_0 on the surface portion of Y . Let ρ_r denote a simple path on the surface starting at x_0 , contained in the plane determined by x_0 and the central axis, with endpoint at distance r from the central axis. Note that a simple loop α_r of constant radius $r > 0$ on the surface is not freely homotopically trivial by [3, Lemma 3.1]. Any neighbourhood of a point of the central axis contains such a loop. For every $1 > r > 0$ loops $\rho_r \alpha_r \rho_r^{-1}$ (with α_r appropriately based) are homotopic to each other and non-trivial. Given a cover of Y , all such loops are contained in the Spanier group of such a cover for sufficiently small r . Hence the Spanier group is non-trivial.

(2) Note that every point $x \in Y$ has an arbitrarily small neighbourhood whose path component containing x is contractible. Given a point on the surface at the radius r , an open ball of radius at most $r/2$ suffices. The interior of the connecting arc C induces such neighbourhoods for points of itself. Any open set not containing the entire arc C suffices for the central axis, e.g. an open ball of radius at most $c/2$ where c is the length of arc C . The claim follows by definition as the loops v_j of Definition 2.5 are contractible. This also proves (3).

(4) The loops α_r from the proof of (1) are non-trivial and arbitrarily close to the central axis.

(5) Implied by (7) and Remark 2.11.

(6) Follows from the proof of (1) using the loops ρ_r .

(7) We use the notation of Definition 2.10 and the fact that every point $x \in Y$ has an arbitrarily small neighbourhood whose path component containing x is contractible. Given any open cover U_1, \dots, U_k of $w([0, 1])$ by such neighbourhoods, the only possible homotopy class for a path $w_1 * \dots * w_k$ is that of w , constructing the homotopy using the contractibility of the sets U_i . Thus the product of the w_i will not be homotopic to v . ■

Proof of Proposition 3.2. (1) The proof of 3.1(1) suffices. Statements (1) and (2) are equivalent as Y' is locally path-connected. Statements (3) and (4) are equivalent for the same reason and follow from (1).

(5) Follows from (7).

(6) Similarly as in proof of 3.1 loops ρ_r provide an obstruction to strong homotopic Hausdorffness.

(7) We will sketch a proof in 3.7. Since the argument is quite lengthy, a complete proof will appear elsewhere. \blacksquare

Proof of Proposition 3.3. Note that every point $x \in Z$ has an arbitrarily small neighbourhood whose every path component is contractible. For any point on the surface at radius r an open ball of radius at most $(1-r)/2$ suffices. The interior of the connecting arc C induces such neighbourhoods for points of itself. Any open set not containing the entire arc C suffices for the outer cylinder, e.g. an open ball of radius at most $c/2$ where c is the length of the arc C . Such cover proves (1)–(7). \blacksquare

Proof of Proposition 3.4. (1) and (2) are equivalent by Theorem 2.8 and imply (3) and (4).

(1) For every positive $r < 1$ let α_r denote a positively oriented simple loop of constant radius on the surface. All such loops are homotopically trivial. Let α_1 be a clockwise oriented simple closed curve, defined as an intersection of the outer cylinder with the plane at height zero. Note that α_1 is not homotopically trivial by [3, Proof of Lemma 3.1]. We will prove that the loop α_1 based at x_0 on the outer cylinder is contained in the Spanier group of Z' .

Given any open cover \mathcal{U} of $\alpha_1([0, 1])$ choose a finite refinement by balls U_0, \dots, U_k so that $U_i \cap U_j \neq \emptyset$ for $i, j \in \mathbb{Z}_{k+1}$ iff $|i - j| \leq 1$. Choose $R < 1$ big enough so that $\alpha_R([0, 1])$ is covered by U_0, \dots, U_k as well. Fix points $b_i \in U_i \cap U_{i-1} \cap \alpha_1$ and $a_i \in U_i \cap U_{i-1} \cap \alpha_R$, so that the pairs (a_i, b_i) are endpoints of the same tunnel. Note that for every $i \in \mathbb{Z}_{k+1}$ an oriented quadrilateral loop Q_i with vertices $[b_i, b_{i+1}, a_{i+1}, a_i]$ (where each edge is an appropriate simple path contained in α_1, α_R or in some tunnel) is contained in U_i as suggested by Figure 6. Define a path c_i between x_0 and b_i to be the restriction of α_1 to the appropriate interval. The loop

$$(c_k \ Q_k \ c_k^{-1}) (c_{k-1} \ Q_{k-1} \ c_{k-1}^{-1}) \dots (c_0 \ Q_0 \ c_0^{-1})$$

of the Spanier group with respect to \mathcal{U} is homotopic to the non-trivial α_1 . Since α_1 does not depend on a cover \mathcal{U} , the based Spanier group of Z' is non-trivial.

(5) Implied by (6) and Remark 2.11.

(6) This is proved in [3, Theorem 3.5].

(7) We adopt the notation of part (1). Let α_1 be a simple loop in the outer cylinder based at x_0 . Assume that x_0 is the endpoint of some tunnel, and that τ_R is the segment of this tunnel of length $1-R$ starting at x_0 . We will only use such R -values where the endpoint of τ_R is an intersection point with the surface portion. We prove that the conditions of Definition 2.10(3) cannot be satisfied for $w = \alpha_1$ and $v = \tau_R * \alpha_R * \tau_R^{-1}$. Assume the opposite. Then, since $w \not\simeq 1$ but $v \simeq 1$ independently of R , there is a finite covering U_1, \dots, U_k of $\alpha_1([0, 1])$ as in Definition 2.10(3). Without loss of generality we can decrease the sets U_i to path-connected sets, so that also the intersections $U_1 \cap U_k$ and $U_i \cap U_{i+1}$ for $i = 1, \dots, k-1$ are path-connected. Let $R < 1$ be big enough so that U_1, \dots, U_k covers $\tau_R * \alpha_R * \tau_R^{-1} = v$. We can reparametrize v so that $v([t_{j-1}, t_j]) \subset U_j, \forall j = 1, \dots, k$. For every $j < k$ define c_j to be a path in $U_j \cap U_{j+1}$ between $w(t_j)$ and $v(t_j)$, and for simplicity let $c_0 = c_k$

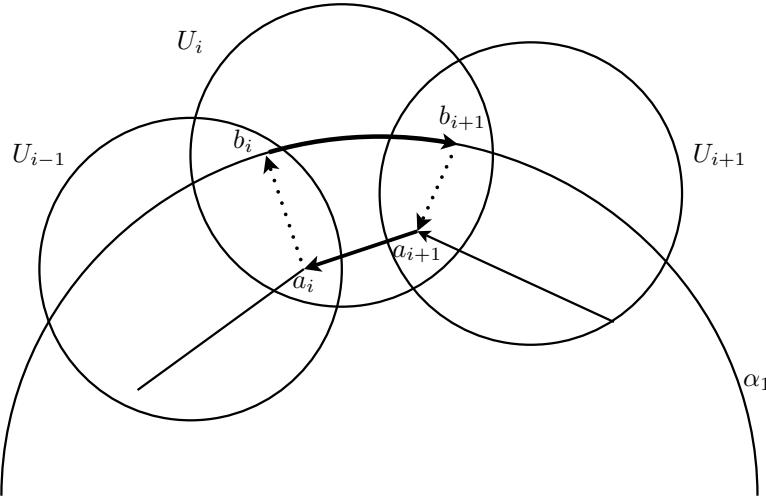


FIGURE 6. Quadrilateral loop Q_i . The pointed lines mark path through the tunnel.

be the constant path at x_0 . Now for $i = 1, \dots, k$ let $w_i := c_{i-1} * v|_{[t_{i-1}, t_i]} * c_i^{-1}$. Obviously $w_1 * \dots * w_k \simeq v$, contradicting the conclusion of Definition 2.10(3). \blacksquare

Definition 3.5. *We call a space X shape injective, if the natural homomorphism $\pi_1(X) \rightarrow \check{\pi}_1(X)$ is injective. For further information see Section 3 of [6].*

Proposition 3.6. *The (based and unbased) Spanier groups of the Hawaiian Earring are trivial.*

Proof. We argue for the unbased Spanier-group, since the based Spanier group is just a subgroup of it, anyhow.

This proposition can be deduced from the literature, namely from the shape-injectivity that the Hawaiian Earring enjoys as a one-dimensional set ([4, Theorem 1.1] or [2, Theorem 5.11]) or as planar set ([5, Theorem 2]), together with the proposition [6, 4.8] which in this case implies that the Spanier group is contained in a kernel of an injective map. \blacksquare

Theorem 3.7. *The space Y' is homotopically path-Hausdorff.*

Sketch of the proof. A space can only fail to fulfill the homotopic path-Hausdorffness, if arbitrarily close to some path w there exist homotopic representatives of the same path v with $v \not\simeq w$. In such a case the space must be wild in the neighbourhood of the trace of w . Now Y' is wild only at the central axis, which is a contractible part of the space. Thus w will be in the trivial homotopy class, and v must be non-trivial. Therefore v must leave the central axis through some tunnel and return in such a way that it cannot be deformed onto the central axis (e.g., by using other tunnels, or since it has circled around the central axis before return). In any case v must have left the tunnel and also spent a segment on the surface. However, the place where v leaves the tunnel and continues via running over the waved surface, will be a characteristic of the non-trivial homotopy class of v . Now, when in order to violate the condition of Definition 2.10(3), we have to construct another homotopic representative of v subordinated to a covering that does not

contain the places mentioned in the previous sentence, it will belong to a different homotopy class. Thus our attempts to violate Definition 2.10(3) are bound to fail.

In a projected forthcoming publication we plan to extend the techniques of combinatorially describing homotopy classes of paths in \mathcal{HE} from [15, 2.3–2.10], [16, Section 1] to the space Y' . Based on the appropriate combinatorial tool we plan to publish a precise proof of this theorem, also.

The situation (cf. Proposition 3.4(7)) was different for the space Z' : Here the wild part contained non-nullhomotopic curves; thus it was possible that $w \not\simeq 1$ and $v \simeq 1$. Indeed all paths $\tau_R * \alpha_R * \tau_R^{-1}$ that we constructed in the proof of Proposition 3.4(7) were nullhomotopic.

4. PROOF OF THEOREM 2.8: ALGEBRAIC CHARACTERIZATION OF SEMILOCAL SIMPLE CONNECTIVITY

(1, “ \Leftarrow ”): If the Spanier group $\pi(\mathcal{U}, x_0)$ vanishes for some covering \mathcal{U} , then the products $u_1 v_1 u_1^{-1}$ of expression (2.3) are nullhomotopic loops. Such a product is contractible if and only if v_1 is contractible as both loops are freely homotopic. Furthermore, the contractibility of a loop never depends on whether the endpoint is kept fixed. Thus the elements of the covering \mathcal{U} suffice to prove that X is unbased semilocally simply connected as every loop v_1 that is contained in some element of \mathcal{U} is contractible.

(1, “ \Rightarrow ”): For every point of X choose a neighbourhood which satisfies the condition of Definition 2.2 and form the covering \mathcal{U} of these neighbourhoods. Every product in the form of expression (2.3) can be contracted by first contracting the v_i -loops, and then contracting $u_i u_i^{-1}$. Hence the Spanier group is trivial.

(2, “ \Leftarrow ”): If for some open covering $\mathcal{V} = \{(U_j, x_j) \mid j \in J\}$ by pointed sets (U_j, x_j) the Spanier group $\pi^*(\mathcal{V}, x_0)$ vanishes, then also the products $u_1 v_1 u_1^{-1}$ of expression (2.5) are nullhomotopic paths. Based on these substitutions, the same line of arguments as in (1, “ \Leftarrow ”) can be used. However, while in expression (2.3) v_i could have been any loop in a neighbourhood U_j , the definition of π^* requires to consider only such v_i that are based at x_j . If U_j should contain loops whose trace has no path-connection to x_j inside U_j , our assumptions do not suffice to conclude that such loops will also be contractible. Thus in this case we can only conclude that Definition 2.1, but not that Definition 2.2 will be fulfilled.

(2, “ \Rightarrow ”): For every point $x \in X$ choose a neighbourhood U_x so that $\pi_1(U_x, x) \rightarrow \pi_1(X, x)$ is zero. Form an open covering \mathcal{U} of pointed sets $\{(U_x, x) \mid x \in X\}$ and consider its based Spanier group. Every product in the form of expression (2.5) can be contracted by first contracting the loops v_i , and then contracting $u_i u_i^{-1}$. Hence the based Spanier group is trivial.

(3): It suffices to observe that the triviality of all loops implies triviality of all based loops within any arbitrary neighbourhood.

(4): Given a point $x \in X$ choose a neighbourhood U of x that satisfies the condition of Definition 2.1. Then any path-connected neighbourhood $V \subset U$ of x satisfies the condition of Definition 2.2.

(5): See Proposition 3.1. ■

5. PROOF OF THEOREM 2.9 : ALGEBRAIC CRITERIA FOR HOMOTOPIC HAUSDORFFNESS

Convention: Within this section the overline (“ $\overline{(\cdot)}$ ”) denotes the reversion of the orientation of a path.

Proof: We split the statement of Theorem 2.9 into two implications:

$\varprojlim \pi^*(\mathcal{U})$ is trivial. $\xrightarrow{(1)}$ homotopic path-Hausdorffness $\xrightarrow{(2)}$ existence of a generalized universal covering space.

(1): Suppose X is not homotopically path-Hausdorff. Using the notation of Definition 2.10(3) there exist paths $w, v: [0, 1] \rightarrow X$ with $w(0)=v(0)=P$, $w(1)=v(1)=Q$ and a non-trivial homotopy class $\alpha \in \pi_1(X, P)$, $\alpha := [w*v]$ for which the conditions of Definition 2.10(3) do not hold. We claim that α is contained in $\varprojlim \pi^*(\mathcal{U})$. The proof will resemble that of Proposition 3.4(1).

Let \mathcal{U} be an open cover of X by pointed sets (cf. Definition 2.4). Choose a cover $U_1, U_2, U_3, \dots, U_k$ of $w([0, 1])$ by open sets from \mathcal{U} so that there exists a partition $0 = t_0 < t_1 < t_2 \dots < t_k = 1$ for which U_j covers $w([t_{j-1}, t_j])$ and the according base point for each of the U_j lies on the segment $w([t_{j-1}, t_j])$. Since Definition 2.10(3) is not satisfied there exist paths w_j such that w_j connects $w(t_{j-1})$ with $w(t_j)$ inside U_j , and so that the concatenation $v' := w_1 * \dots * w_k$ is homotopic to v . Assume that v' is parametrized so that $v(t_j) = w(t_j)$ for $j = 0, \dots, k$. Note that the concatenation

$$(v|_{[t_0, t_0]} * w|_{[t_0, t_1]} * \overline{v|_{[t_0, t_1]}}) * (v|_{[t_0, t_1]} * w|_{[t_1, t_2]} * \overline{v|_{[t_0, t_2]}}) * \dots \\ \dots * (v|_{[t_0, t_{k-1}]} * w|_{[t_{k-1}, t_k]} * \overline{v|_{[t_0, t_k]}})$$

is homotopic to α and contained in $\pi^*(\mathcal{U}, P)$ as each of the factors $v|_{[t_0, t_{i-1}]} * w|_{[t_{i-1}, t_i]} * \overline{v|_{[t_0, t_i]}} = v|_{[t_0, t_{i-1}]} * w|_{[t_{i-1}, t_i]} * \overline{v|_{[t_{i-1}, t_i]}} * \overline{v|_{[t_0, t_{i-1}]}}$ is a conjugate of the loop $w|_{[t_{i-1}, t_i]} * \overline{v|_{[t_{i-1}, t_i]}}$, that is contained in $U_j \in \mathcal{U}$ and has its base point $w(t_{i-1})$ inside U_j path-connected to the base point of U_j .

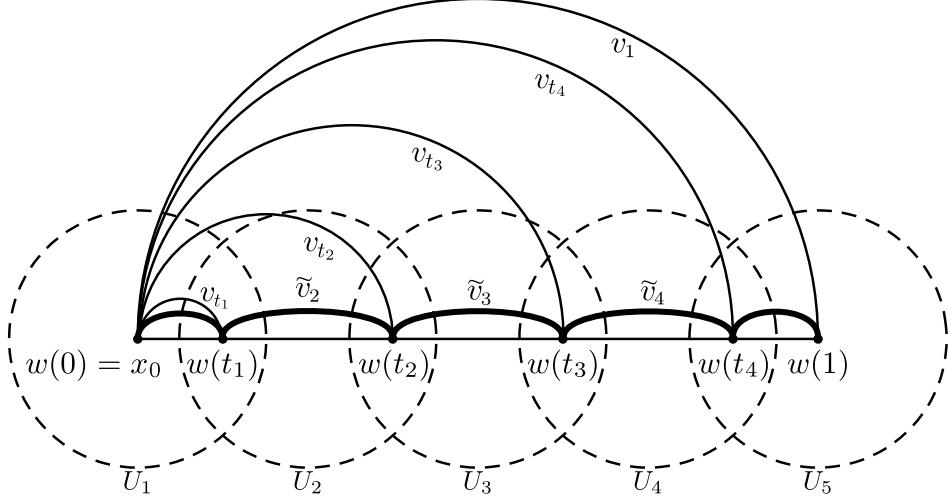
(2): We give a proof by contradiction. Recall that the standard existence proof of covering spaces (e.g. [11, p. 393]) is based on interpreting the Universal Path Space (as it was called by [1]) as a covering space. Also, the generalized universal covering spaces in the sense of [6] are constructed on the basis of considering the Universal Path Space, with an adaptation of the definition of the topology to the situation of the absence of semilocal simple connectivity. Therefore the points in the covering space of (X, x_0) are represented by (homotopy classes of) paths in the base space and the topology is induced by the sets

$$\mathbb{U}(\gamma, U) = \{\gamma * \delta; \delta: [0, 1] \rightarrow U, \delta(0) = \gamma(1)\}$$

where $\gamma: [0, 1] \rightarrow X$ is a path originating at x_0 and $U \subset X$ is an open neighbourhood of $\gamma(1)$.

The assumption for the desired proof by contradiction is the following: There does not exist a generalized universal covering space. By [6, 2.14] this means, that there exists a path $w: [0, 1] \rightarrow X$ which allows two different lifts to the covering space with the same start-point. We can assume that $t \mapsto w_t := w|_{[0, t]}$ and $t \mapsto v_t$ are two different lifts of path w with $v_0 = w_0$ being a constant path at $w(0)$ and $v := v_1 \not\simeq w_1 = w$. We claim that such a situation does not comply with the conditions of Definition 2.10(3).

Choose any covering U_1, \dots, U_k of $w([0, 1])$ such that for a suitable partition $0 = t_0 < t_1 < t_2 < t_3 < \dots < t_k = 1$, U_j covers $w([t_{j-1}, t_j])$, as suggested by Figure 7. Let us focus on the interval $[t_{i-1}, t_i]$ for some fixed i . By the continuity of the lift v there exists for every $t \in [t_{i-1}, t_i]$ a neighbourhood V_t of t so that $v_s \in \mathbb{U}(v_t, U_i), \forall s \in V_t$. Hence we can find a partition $t_{i-1} = s_0 < s_1 < \dots < s_l = t_i$ so that $v_s \in \mathbb{U}(v_{s_j}, U_i), \forall s \in [s_j, s_{j+1}]$, i.e. $v_{s_j} * \delta_j \simeq v_{s_{j+1}}$ for some path δ_j in U_i . This condition implies the existence of a path $\tilde{v}_i: [t_{i-1}, t_i] \rightarrow U_i$ between $v_{t_{i-1}}(1)$ and $v_{t_i}(1)$, so that $v_{t_{i-1}} * \tilde{v}_i \simeq v_{t_i}$. The paths \tilde{v}_i , defined on $[t_{i-1}, t_i]$, induce a path $\tilde{v}: [0, 1] \rightarrow X$.

FIGURE 7. Two different lifts of the path w .

The drawing illustrates the proof-construction of step (2) in the special situation where we have $k=5$. In addition, this figure is drawn in a way that for each j we just get $l=1$, i.e. just the case of a trivial s -partition is pictured. The line in bold corresponds to the path \tilde{v} .

Note that

$$\begin{aligned} \tilde{v} &:= \tilde{v}_1 * \tilde{v}_2 * \dots * \tilde{v}_k \simeq (\overline{v_0} * v_0) * \tilde{v}_1 * (\overline{v_{t_1}} * v_{t_1}) * \tilde{v}_2 * \dots * (\overline{v_{t_{k-1}}} * v_{t_{k-1}}) * \tilde{v}_k \simeq \\ &\simeq (v_0 * \tilde{v}_1) * \overline{v_{t_1}} * (v_{t_1} * \tilde{v}_2) * \overline{v_{t_2}} * (v_{t_2} * \tilde{v}_3) * \dots * \overline{v_{t_{k-1}}} * (v_{t_{k-1}} * \tilde{v}_k) \simeq \\ &\simeq v_{t_1} * \overline{v_{t_1}} * v_{t_2} * \overline{v_{t_2}} * v_{t_3} * \dots * \overline{v_{t_{k-1}}} * v_{t_k} \simeq v_{t_k} = v_1 = v \end{aligned}$$

hence with letting $w_i := \tilde{v}_i$ we obtain that Definition 2.10(3) is not fulfilled. \blacksquare

6. OVERVIEW OF THE IMPLICATIONS

The diagram of this section (see Fig. 8) gathers together a number of implications of properties of a space that occurred in, or are closely related to, the ones from our Theorems 2.8 and 2.9. No assumption other than path-connectedness (cf. Remark 2.7) is made here. According to the enumeration of the implications in the diagram, for each arrow a reference or a sketch of the proof is given. The label (1, “ \Rightarrow ”) means, that an argument is to be given, why this implication is true, while (1, “ \Leftarrow / ”) means, that an argument is to be given, why the converse of this implication is in general not true.

(1, “ \Rightarrow ”): Follows from the definition of π_1 .

(1, “ \Leftarrow / ”): The counterexample is Y (see Proposition 3.1).

(2, “ \Rightarrow ”): This is a passage to an obviously weaker property.

(2, “ \Leftarrow / ”): The Hawaiian Earring is an appropriate example. It contains arbitrarily small essential loops in the neighbourhood of the accumulation point. Therefore it is not semilocally simply connected. On the other hand, one-dimensional spaces are weakly homotopically Hausdorff by [2, Corollary 5.4(2)].

(3, “ \Rightarrow ”): The assumption means that every point has a neighbourhood such that all loops in this neighbourhood are contractible. Such neighbourhoods suffice to prove that the space is strongly homotopically Hausdorff.

(3, “ \Leftarrow / ”): Similarly as for (2, “ \Leftarrow / ”) the Hawaiian Earring is an appropriate example. It is easy to see that it is not unbased semilocally simply connected at

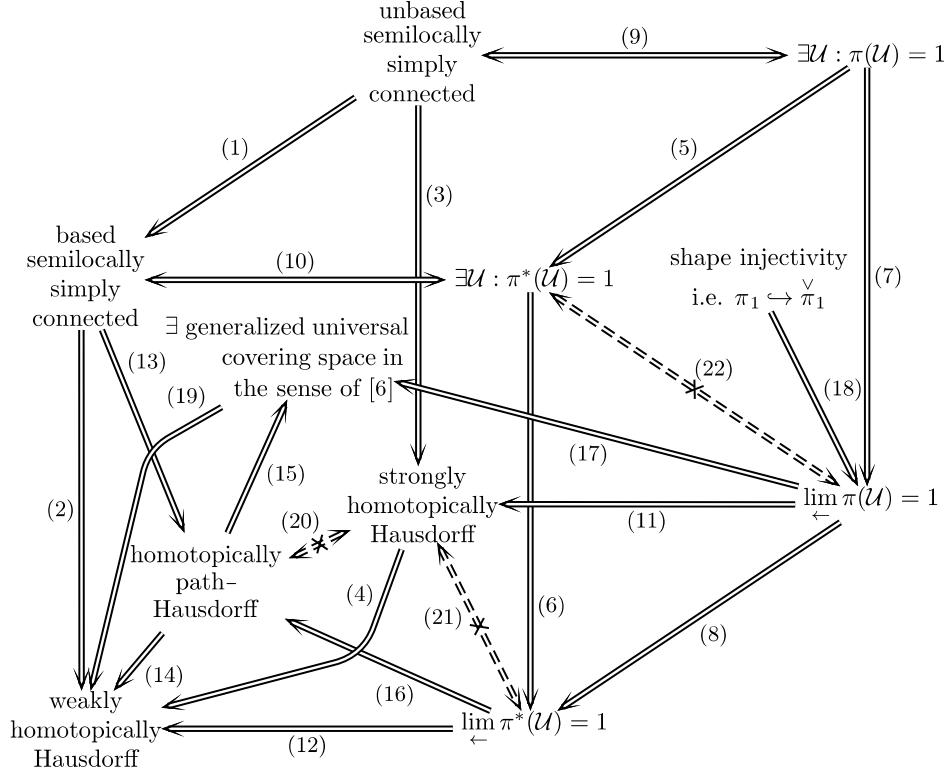


FIGURE 8. Diagram

the accumulation point due to arbitrarily small essential loops. It is slightly harder to see that it is strongly homotopically Hausdorff.

Let us denote the accumulation point of the Hawaiian Earring HE by x_0 and let us enumerate the loops of HE by $l_i, i = 1, 2, \dots$, i.e. $\cap_i l_i = \{x_0\}$ (cf. Figure 5). It is enough to consider the condition for strong homotopic Hausdorffness at x_0 as all other points have contractible neighbourhoods. Let $\{V_i\}_{i=1,2,\dots}$ denote a basis of open neighbourhoods of x_0 so that $l_i \subset V_j$ iff $i \geq j$. Suppose there exists a loop $\alpha = \alpha_1$ which is freely homotopic to some loop α_i in V_i for every $i > 1$. Note that there is a strong deformation retraction

$$V_i \rightarrow W_i := \bigcup_{j \geq i} l_j.$$

Therefore we can assume that every α_i is a loop in W_i . For every i the loops α_i and α_{i+1} are homotopic. By Lemma 4.3(1) of [3] the homotopy h_i between them can be chosen within W_i . We define a map $f: B^2(o, 1) \rightarrow HE$ on a closed unit disc with midpoint o by assigning $f|_{S^1(o, 1/i)} = \alpha_i$ and by using the homotopies h_i for defining

$$f|_{\overline{B^2(o, 1/i)} - B^2(o, 1/(i+1))} := h_i$$

on the closed annuli between the concentric circles $S^1(o, 1/i)$. The map f is obviously continuous on $B^2(o, 1) - \{o\}$ as it is obtained by a locally finite gluing of maps that agree on the intersections of annular domains (i.e. on concentric circles). It is also continuous at o because for every i the preimage of V_i contains $B^2(o, 1/i)$. The map f provides a nullhomotopy for $\alpha = \alpha_i$.

(4, \Rightarrow): This is a passage to a weaker requirement.

(4, \Leftarrow): The space Y serves as an example by Proposition 3.1.

(5, “ \implies ”): This is a weakening of the conditions, since π^* -groups contain fewer elements than π -groups.

(5, “ \nLeftarrow ”): The space Y serves as an example. To prove it we use Proposition 3.1. The non-triviality of the Spanier group of Y and (7) assure that there is no cover with non-trivial Spanier group. Yet Y is semilocally simply connected.

(6, “ \implies ”): By Remark 2.6 the condition “ $\pi^*(\mathcal{U}) = 1$ ” for some cover \mathcal{U} guarantees the triviality of the inverse limit.

(6, “ \nLeftarrow ”): Again the Hawaiian Earring gives the corresponding example. Every covering of the accumulation point contains small essential loops entirely and thus no $\pi^*(\mathcal{U})$ -group can be trivial. In Proposition 3.6 it was shown that it has trivial Spanier group.

(7, “ \implies ”): The argument here is analogous to that of (6, “ \implies ”).

(7, “ \nLeftarrow ”): The same example as for (6, “ \nLeftarrow ”) suffices. For locally path-connected spaces there is no difference between the based and unbased statements.

(8, “ \implies ”): Follows from the same argument as (5, “ \implies ”).

(8, “ \nLeftarrow ”): The space Y can be used for this purpose by Proposition 3.1.

(9, “ \iff ”): This is the statement of Theorem 2.8(1).

(10, “ \iff ”): This is the statement of Theorem 2.8(2).

(11, “ \Leftarrow ”): If a space is not strongly homotopically Hausdorff, then there exists a point P and a non-trivial free homotopy class α which can be represented in any neighbourhood of P . This class will appear in every $\pi(\mathcal{U})$ (and thus in their inverse limit), since each \mathcal{U} has to contain a neighbourhood of P .

(11, “ \nRightarrow ”): The corresponding example is Z' . By Proposition 3.4 it is strongly homotopically Hausdorff but has a non-trivial Spanier group.

(12, “ \Leftarrow ”): The same argument as for (11) applies in terms of based homotopies.

(12, “ \nRightarrow ”): The same counterexample as for (11) applies as the space Z' is locally path-connected, thus the base points do not matter.

(13, “ \implies ”): The implication follows from statements (10), (6) and (16).

(13, “ \nLeftarrow ”): The Hawaiian Earring can be used as an example. It is obviously not semilocally simply connected. On the other hand, the proof of (6, “ \nLeftarrow ”) establishes the triviality of the based Spanier group of HE , hence it is homotopically path-Hausdorff by (16).

(14, “ \implies ”): This has been observed in Remark 2.11 already.

(14, “ \nLeftarrow ”): The space Z' can be used for this purpose by Proposition 3.4.

(15, “ \implies ”): This is implication (2) from the proof of Theorem 2.9 in Section 5.

(16, “ \implies ”): This is implication (1) from the proof of Theorem 2.9 in Section 5.

(16, “ \nLeftarrow ”): The corresponding example is Y' by Proposition 3.2.

(17, “ \implies ”) and (18, “ \implies ”): These implications have been proved in [6, 4.7] and [6, 4.8] respectively. Implication (17) also follows from implications (8),(16) and (15).

(17, “ \nLeftarrow ”): The corresponding example is Y' : By Proposition 3.2(1) it has non-trivial Spanier group, but according to (15) and Proposition 3.2(7) it has a generalized covering space.

(18, “ \nLeftarrow ”): The corresponding example is Z . The fundamental group of Z is \mathbb{Z} generated by the outer cylinder. This follows, since by [3, Lemma 4.3(1)] any nullhomotopy of a loop that is not passing through the arc C need not to pass through C , either, but $Z - C$ consists of two different path-components: the outer cylinder, and the contractible surface portion. The fundamental group of Z vanishes when we pass to the shape group (i.e. the outer cylinder is homotopically trivial in every neighbourhood of Z in \mathbb{R}^3), hence the space is not shape injective. On the other hand the Spanier group of Z is trivial by Proposition 3.3(1).

(19, “ \Leftarrow ”): By contradiction: The presence of small loops in the sense of [13] is equivalent to the absence of weak homotopic Hausdorffness. Every small loop induces a non-trivial lift by (cf. [13, Lemma 16] or [6, Lemmas 2.10–2.11]), hence there is no generalized universal covering space in terms of [6].

(20, “ $\nLeftarrow \nRightarrow$ ”): The corresponding examples are Y and Z' by Propositions 3.1 and 3.4.

(21, “ $\nLeftarrow \nRightarrow$ ”): The corresponding examples are Y and Z' by Propositions 3.1 and 3.4.

(22, “ \nRightarrow ”): The corresponding example is Y by Proposition 3.1(1) and the proof of Proposition 3.1(2).

(22, “ \Leftarrow ”): The corresponding example is the Hawaiian Earring. The argument is the same as that of “(6 \Leftarrow)”, since there is no difference between based and unbased statements for locally path-connected spaces. \blacksquare

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